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Critical Percolation on the Torus

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We compute the various crossing probabilities defined by R. Langlands, P. Pouliot, and Y. Saint-Aubin for the critical percolation on the torus.

KEY WORDS: Percolation; crossing probability; torus.

Langlands *et al.* (LPS)[5] defined a certain crossing probability for the critical percolation on any Riemann surface. Define a lattice on the Riemann surface as a pullback of the standard square lattice on the complex plane by a meromorphic function on the Riemann surface. At each bond (or site), assign a certain probability of being open. Then for each configuration s of open bonds (sites) there is a collection X_s consisting of clusters of open bonds (or sites). The homology group $H(X_s)$ with integer coefficients can then be linearly mapped into the homology group $H(R)$ of the Riemann surface. Then we can ask for the probability that a certain subgroup of $H(R)$ will lie in the image of some linear map. They expect the resulting probability to be conformally invariant and independent of the meromorphic map.

In a related work, Cardy[2] formulated the probability of a crossing between two curves on the boundary of a rectangle in the critical percolation as a correlation function of the Q-state Potts model as $Q \to 1$. Using the techniques of conformal field theory, he computed the crossing probability.

In related work, di Francesco *et al.*[3] computed the partition function of the Q-state Potts model on the torus by reducing it to calculations involving the Gaussian functional integral. They first formulated the Q-state Potts model as a six-vertex model. Since this particular six-vertex

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model, namely the F-model, renormalizes to a Gaussian free field, they were able to compute the partition function on the torus. Although they only calculate the partition function by summing over all configurations, it turns out that their analysis can be adapted to summations over a restricted class of configurations which is required to compute the various LPS crossing probabilities on the torus. In this way we evaluate the LPS crossing probabilities with respect to all subgroups of the homology group of the torus at the continuum limit [Eq. (3.16)]. However, we compute the LPS crossing probability with respect to a lattice different from the pullback lattice as defined above. A natural lattice reflecting the complex structure exists on the torus at the continuum limit (Section 2). The results are in excellent agreement with the numerical data obtained by Langlands *et al.* Moreover, as expected, the LPS crossing probabilities are modular invariant.

In fact we feel that this technique can be used to evaluate the LPS crossing probability on all Riemann surface with genus $g > 1$ since all computations reduce essentially to evaluations of some Gaussian functional integral. We are currently investigating this.

1. GAUSSIAN FREE FIELD THEORY

The action of the Gaussian free field theory is given by

$$\mathscr{A} = \frac{g}{4\pi} \int_T |\nabla\varphi|^2 \, d^2x \tag{1.1}$$

where the integral is taken over the torus T. A Gaussian functional integral is then given by

$$Z_{m',m}(g) = \int_{\substack{\delta_1\varphi = 2\pi m \\ \delta_2\varphi = 2\pi m'}} D\varphi \, e^{-\mathscr{A}} \tag{1.2}$$

where the multivalued function φ picks up an additive constant $2\pi m$ ($2\pi m'$) as φ is transported along the cycle 1 (τ). Using the ζ-function regularization (see, for example, ref. 4), we find that the functional integral reduces to

$$Z_{m',m}(g) = \frac{\sqrt{g}}{\tau_I^{1/2}\eta(q)\,\bar{\eta}(q)} \exp\left\{ -\frac{\pi g}{\tau_I} \left[m'^2 + m^2(\tau_I^2 + \tau_R^2) - 2\tau_R m m' \right] \right\} \tag{1.3}$$

where the Dedekind eta function $\eta(q)$ is given by

$$\eta(q) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k), \qquad q = e^{2\pi i\tau} \tag{1.4}$$

A modular invariant Coulomb partition can be constructed by summing over m', m:

$$Z_c[g, f] = f \sum_{m', m \in fZ} Z_{m'm}(g) \tag{1.5}$$

2. LATTICE ON THE TORUS

Langlands *et al.* define a percolation on a Riemann surface by taking any meromorphic function from the Riemann surface to the complex plane and puling back the square lattice from the plane to the Riemann surface.

On the torus we can find a lattice (at least at the continuum limit) in another way. Take a cylinder with circumference 1 and some height h. If the height h is 1, then we can put n squares along the circumference and n squares along the height and identify the top and bottom of the cylinder. We then take the continuum limit $n \to \infty$. If the height h is some irrational number, then we can approximate it by a sequence of rational numbers m_i/n_i as $i \to \infty$. Put n_i squares along the circumference and m_i squares along the height and identify the top and the bottom of the cylinder. We then take the continuum limit as $i \to \infty$. Have we exhausted all possibilities? We can give a different identification at the top and the bottom by twisting a certain length θ. If we go a length one, then we have made a complete turn. To make a twist of irrational length θ, we make a rational approximation as in the case of the irrational height and take the continuum limit. In this way we have exhausted all possible tori by taking all possible (h, θ). In the standard discussion all tori are realized as the complex plane moded out by a lattice $C/Z + Z\tau$. The two descriptions are related by the equation $\tau = \theta + ih$. This construction can be realized on higher compact Riemann surfaces, and we are currently formulating this.

3. PERCOLATION ON THE TORUS

We adapt the techniques used by di Francesco *et al.*[3] to compute all the LPS crossing probabilities on the torus with respect to the lattice described above at the continuum limit.

Since the percolation probabilities can be written in terms of the partition function of the Q-state Potts model,[1,5] we first describe the Q-state Potts model on the torus. Let there be a lattice L on a torus such that there are Q possible values at each site. Then the partition function is the summation over some class of configurations

$$\mathscr{L}_Q = \sum \exp\left[-\frac{1}{T} \sum \delta(q_i, q_j) \right] \tag{3.1}$$

where the delta function is 0 if the nearest neighbor pair differs in value and 1 otherwise. Letting $\mathcal{N}_{\mathcal{B}}$ be the number of bonds in all the clusters and $\mathcal{N}_{\mathcal{C}}$ be the number of connected components (a connected component can be either an isolated point or a cluster of bonds), it is well known[1,5] that this expression can be rewritten as

$$\mathcal{L}_Q = \sum (1 - e^{-1/T})^{\mathcal{N}_{\mathcal{B}}} Q^{\mathcal{N}_{\mathcal{C}}} \tag{3.2}$$

By restricting the sum in Eq. (3.2) to some class of configurations consisting of clusters of bonds and isolated points, all the LPS crossing probabilities can be realized. In the percolation theory, a collection of all open bonds X_s with respect to a configuration s gives rise to a linear map $H(X_s) \to H(T)$ of homology groups with integer coefficients. The probability that a given subgroup G lies in an image of some map is denoted by $\pi(G)$. A given subgroup must be $\{0\}$, $Z \times Z$, or a subgroup generated by $(a, b) \in Z \times Z$. It is not difficult to see that a and b must be relatively prime if it is to have a nonzero probability.

Thus,

$$\pi(0) = \sum_{\text{clusters homotopic to a point}} (1 - e^{-1/T})^{\mathcal{N}_{\mathcal{B}}} Q^{\mathcal{N}_{\mathcal{C}}} \tag{3.3}$$

$$\pi(Z \times Z) = \sum_{\text{clusters that have cross topology}} (1 - e^{-1/T})^{\mathcal{N}_{\mathcal{B}}} Q^{\mathcal{N}_{\mathcal{C}}} \tag{3.4}$$

$$\pi(a, b) = \sum_{\substack{\text{non-self-intersecting clusters that wrap} \\ \text{around the two cycles } a \text{ and } b \text{ times}}} (1 - e^{-1/T})^{\mathcal{N}_{\mathcal{B}}} Q^{\mathcal{N}_{\mathcal{C}}} \tag{3.5}$$

as $Q \to 1$. The configurations that arises in the sum (3.3) have clusters which are homotopic to a point. In other words, the clusters can be continuously deformed to a point. The configurations that arises in the sum (3.4) must contain a cluster of the cross topology type. These clusters are formed from two independent cycles which have a nontrivial intersection. In the case $a = 1$ and $b = -1$ in the sum (3.5), the configurations must contain a cluster which goes once around each cycle. We have chosen the sign convention of a and b so that going in the positive direction along 1 is positive and going in the negative τ direction is positive when the torus is viewed as $C/Z + Z\tau$. See Figs. 1 and 2.

To make connections with the Gaussian free field theory, the partition functions of the Q-state Potts model must be formulated as a six-vertex model.[1] We discuss the relevant features of this transformation. We first form another lattice L' by connecting the four midpoints of the four bonds of a face of the original lattice L. Thus another square lattice L' results. A polygonal decomposition can be made by splitting each midpoint to form either two disjoint squares or an eight-sided polygon. A cluster then corresponds to a long polygon formed by the bonds of the new lattice L'. Letting $\mathcal{N}_{\mathcal{S}}$ be the number of sites in the original lattice L and $\mathcal{N}_{\mathcal{P}}$ the number of polygons (note that an isolated point corresponds to a square), we have the following topological facts. If a configuration has a cross topology, the Euler relation

$$\mathcal{N}_{\mathcal{P}} = \mathcal{N}_{\mathcal{B}} + 2\mathcal{N}_{\mathcal{C}} - \mathcal{N}_{\mathcal{S}} - 2 \tag{3.6}$$

holds. The configuration in Fig. 3 contains a cluster of the cross topology type. We can directly compute $\mathcal{N}_{\mathcal{B}} = 28$, $\mathcal{N}_{\mathcal{C}} = 38$, $\mathcal{N}_{\mathcal{S}} = 64$, and $\mathcal{N}_{\mathcal{P}} = 38$, thus confirming the validity of the formula (3.6) in this instance. See ref. 3 for more details. Otherwise we have the standard Euler relation[1]

$$\mathcal{N}_{\mathcal{P}} = \mathcal{N}_{\mathcal{B}} + 2\mathcal{N}_{\mathcal{C}} - \mathcal{N}_{\mathcal{S}} \tag{3.7}$$

Inserting these relations into (3.3)–(3.5), we generically get

$$\mathcal{L}_{\mathcal{Q}} = Q^{\mathcal{N}_{\mathcal{S}}/2} \sum [(1 - e^{-1/T}) Q^{-1/2}]^{\mathcal{N}_{\mathcal{B}}} Q^{\mathcal{N}_{\mathcal{P}}/2} \tag{3.8}$$

A direction can be assigned to each polygon in a polygonal decomposition by putting arrows on the bonds of the new lattice L'. A configura-

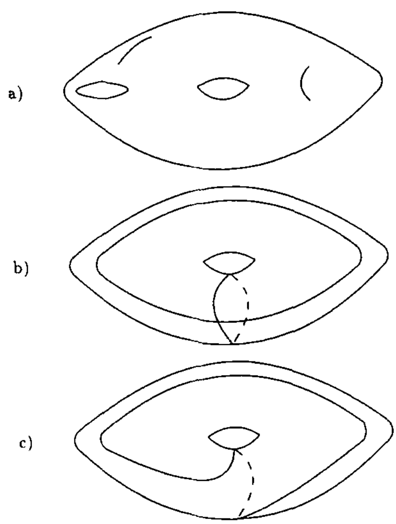

Fig. 1. (a) The configurations in Eq. (3.3) have clusters homotopic to a point. (b) The configurations in Eq. (3.4) must contain a cluster of the cross topology type. (c) The configurations in Eq. (3.5) must contain a cluster that goes around both cycles once if we, for example, let $a = 1$ and $b = -1$.

tion then consists of arrows placed on the new lattice such that two arrows come in and out at any site. If each of the six types of the possible arrow configurations at a site is given a certain weight,[1] the partition functions can be formulated as a summation over all arrow configurations. Since the transformation from the Q-state Potts model to the six-vertex model is discussed at length in ref. 1, p. 323, we will omit it.

Alternatively, the six-vertex model can be formulated[6] by putting an integer multiple of $\pi/2$ at each face of the new lattice L' such that the values on two neighboring faces differ by $\pm\pi/2$. A particular arrow configuration of the six-vertex model can be reproduced by putting a higher value on the left of each arrow.

The restricted summations in Eqs. (3.3)–(3.5) turn into summations over certain boundary conditions due to a topological consequence.[3] A configuration having $\sum \varepsilon_i$ oriented polygons wrapping the torus a times along cycle 1 and b times along cycle τ has a boundary condition

$$\begin{aligned} \delta_1 \varphi &= \frac{\pi}{2} b \sum \varepsilon_i \\ \delta_2 \varphi &= \frac{\pi}{2} a \sum \varepsilon_i \end{aligned} \tag{3.9}$$

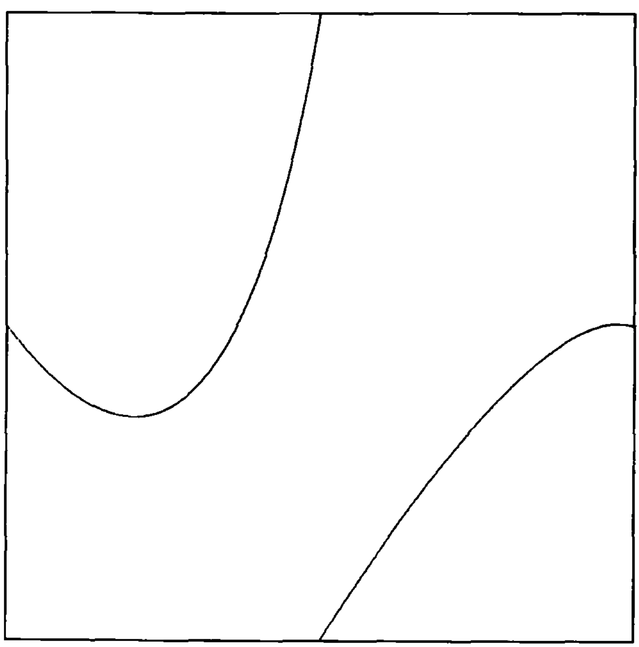

Fig. 2. We choose the sign convention so that this curve (which is the same curve as the one in Fig. 1(c) on the torus is described as $a = 1$ and $b = -1$.

The numbers a and b must be relatively prime and ε_i is ± 1, depending on the orientation of the polygon. Each of these polygons does not intersect the others. We illustrate this in one instance. Consider the configuration given in Fig. 4. This contains a cluster that goes around both cycles once. This cluster can be labeled as $a = 1$ and $b = -1$. Corresponding to this cluster there are two oriented polygons (meaning that a direction has been assigned to them). Since the two polygons have the same directions, they have the same signs. (The signs can be determined in one configuration, and the same convention can be used for all subsequent configurations.) In the present case, we have $\varepsilon_1 = +1$ and $\varepsilon_2 = +1$. Thus, using formula (3.9), we find that the boundary values change by $\delta_1 \varphi = (\pi/2)(-1)(2) = -\pi$ and $\delta_2 \varphi = (\pi/2)(1)(2) = \pi$. We can easily check that an assignment of $\pm \pi/2$ with respect to this configuration leads to a difference in the boundary values as predicted by $\delta_1 \varphi = -\pi$ and $\delta_2 \varphi = \pi$.

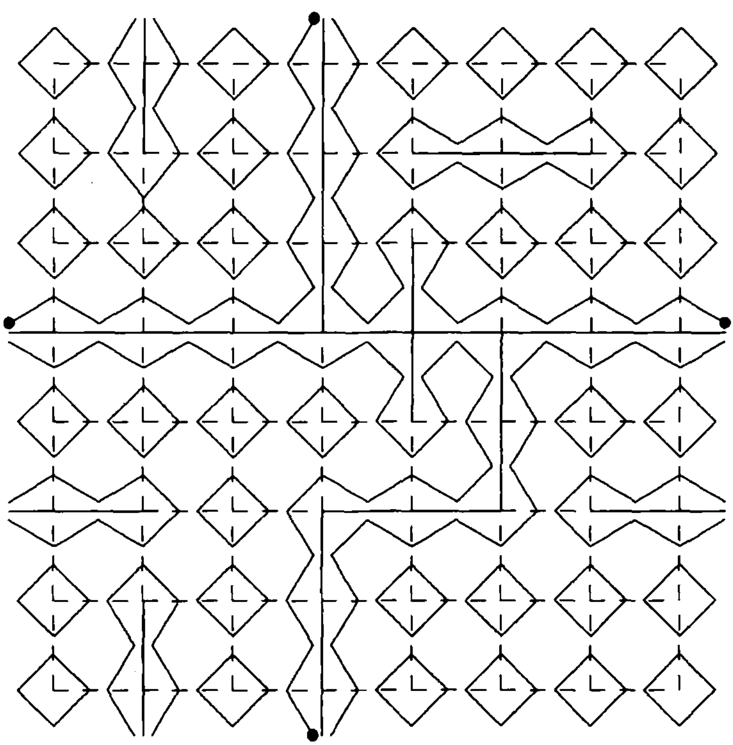

Fig. 3. The dots on the opposite sides of the torus are identified as the same points. This configuration yields $\mathcal{N}_{\mathscr{C}} = 38$, $\mathcal{N}_{\mathscr{S}} = 64$, $\mathcal{N}_{\mathscr{P}} = 38$, and $\mathcal{N}_{\mathscr{B}} = 28$, thus confirming the formula (3.6) in this case.

Fix a boundary condition $\delta_1 \varphi = \pi m$ and $\delta_2 \varphi = \pi m'$. Assume either m or m' is not zero. This means that the sum $\sum \varepsilon_i$ in (3.9) is nonzero. Since each ε_i counts an oriented polygon not homotopic to a point, there must be an oriented polygon not homotopic to a point. Then an oriented polygon not homotopic to a point is present. The Gaussian expression

$$Z_{m'm}\left(\frac{g}{4}\right) \tag{3.10}$$

where the coupling constant g is given by[6]

$$Q = 2 + 2\cos(\pi g/2), \qquad g \in [2, 4] \tag{3.11}$$

has a factor of 1 for each of these polygons while it should have a factor of $e^{\pm 2\pi\lambda}$, where $Q^{1/2} = e^{-2\pi\lambda} + e^{2\pi\lambda}$. To see why this is true, first recall that

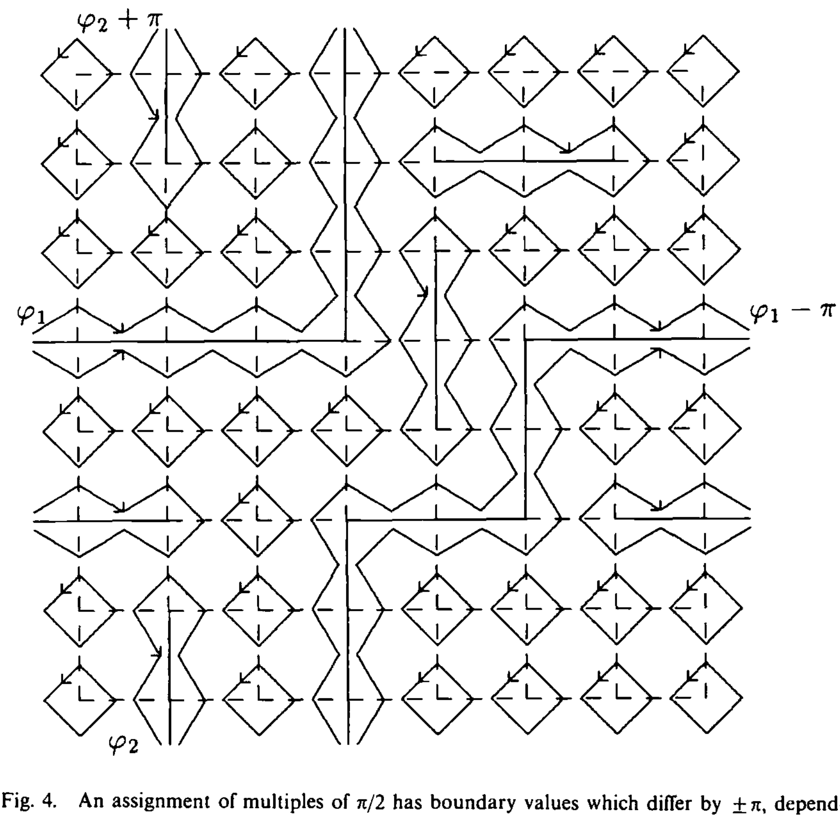

Fig. 4. An assignment of multiples of $\pi/2$ has boundary values which differ by $\pm\pi$, depending on which way we travel.

the transformation from the Q-state Potts model to the six-vertex models introduces a signed angle (ref. 1, p. 328). In the case of these polygons the total signed angle is 0 and not $\pm 2\pi$. Thus we get an incorrect factor of $1 = e^{\pm 0\lambda}$ instead of the correct factor $e^{\pm 2\pi\lambda}$.

Also if $m' = m = 0$, polygons which have cross topology are present. They satisfy a different Euler relation (3.6). Thus the Gaussian expression (3.10) has an incorrect factor of 1 instead of Q for each of these polygons. However, this problem can be ignored because we take the limit $Q \to 1$ in the percolation case.

We remark that the Gaussian expression given by (3.10) and (1.1) is assumed to describe the continuum limit behavior. Since this assumption is based on the renormalization group arguments,[6] it is not rigorously true (in the mathematical sense). However, as we will see, the agreement between computer simulation and the values predicted by the Gaussian expression certainly strengthens the validity of this particular renormalization group argument.

The first defect can be remedied by introducing some factors. Let $Q^{1/2} = 2\cos[(\pi/2)\, e_0]$ and a boundary condition be given by $\delta_1 \varphi = \pi m = (\pi/2)\, b \sum_{i=1}^{k} \varepsilon_i$ and $\delta_2 \varphi = \pi m' = (\pi/2)\, a \sum_{i=1}^{k} \varepsilon_i$. Since a and b are relatively prime (denoted $a \wedge b = 1$, where $a \wedge b$ is the greatest common factor), note

$$\begin{aligned}
&\sum_{\varepsilon_i = \pm 1} \cos[\pi e_0 (m' \wedge m)] \\
&\quad = \sum_{\varepsilon_i = \pm 1} \cos\left(\frac{\pi}{2} e_0 \sum_{i=1}^{k} \varepsilon_i\right) \\
&\quad = \frac{1}{2} \sum_{\varepsilon_i = \pm 1} \left[\exp\left(i \frac{\pi}{2} e_0 \sum_{i=1}^{k} \varepsilon_i\right) + \exp\left(-i \frac{\pi}{2} \varepsilon_0 \sum_{i=1}^{k} \varepsilon_i\right)\right] \\
&\quad = Q^{k/2}
\end{aligned} \tag{3.12}$$

Thus, the correct factor Q^k can be reintroduced by putting in the cosine factors

$$\begin{aligned}
\pi(a, b) = &\sum_{m' = al, m = bl, l \in Z} Z_{m'm}\left(\frac{g}{4}\right) \cos[\pi e_0 (m' \wedge m)] \\
&+ \sum_{\substack{m' \neq al \text{ or } m \neq bl \\ \text{for any } l \in Z}} Z_{m'm}\left(\frac{g}{4}\right) \cos[\pi (m' \wedge m)] \\
&- \sum_{m', m \in Z} Z_{m'm}\left(\frac{g}{4}\right) \cos[\pi (m' \wedge m)]
\end{aligned} \tag{3.13}$$

as $Q \to 1$. Roughly, the first sum contains terms arising from configurations which contain clusters generating the group elements in (a, b). The second sum sets the terms arising from configurations which contain clusters not homotopic to a point and not of the cross topology type equal to zero by setting $e_0 = 1$. The third sum subtracts terms arising from configurations which have clusters with cross topology or homotopic to a point. Finally

$$\pi(0) = \frac{1}{2} \sum_{m', m \in Z} Z_{m'm}\left(\frac{g}{4}\right) \cos[\pi(m' \wedge m)] \tag{3.14}$$

By setting $e_0 = 1$, all contributions from configurations which have clusters not homotopic to a point and not of the cross topology type vanish. The factor 1/2 gets rid of the redundancy created by duality. Also by duality,

$$\pi(Z \times Z) = \pi(0) \tag{3.15}$$

Using explicit formulas

$$\begin{aligned} \pi(a, b) = {} & \sum_{l \in Z} Z_{a3l, b3l}(\tfrac{2}{3}) - \tfrac{1}{2} \sum_{l \in Z} Z_{a(3l+1), b(3l+1)}(\tfrac{2}{3}) - \tfrac{1}{2} \sum_{l \in Z} Z_{a(3l+2), b(3l+2)}(\tfrac{2}{3}) \\ & - \sum_{l \in Z} Z_{a2l, b2l}(\tfrac{2}{3}) + \sum_{l \in Z} Z_{a(2l+1), b(2l+1)}(\tfrac{2}{3}) \\ \pi(0) = {} & \pi(Z \times Z) = \tfrac{1}{2}(Z_c[\tfrac{8}{3}, 1] - Z_c[\tfrac{8}{3}, \tfrac{1}{2}]) \end{aligned} \tag{3.16}$$

Langlands *et al.* derived some numerical values for these crossing probabilities at $\tau = i$. The comparison between the theoretical values and the numerical values is given Table I.

Finally, under modular transformations, i.e., $\tau \to \tau + 1$, $\tau \to -1/\tau$,

$$\begin{aligned} \pi(a, b)|_{-1/\tau} &= \pi(b, -a)|_{\tau} \\ \pi(a, b)|_{\tau+1} &= \pi(a-b, b)|_{\tau} \end{aligned} \tag{3.17}$$

Table I. Comparison Between Numerical and Theoretical Values

Probability	Numerical values	Theoretical values
$\pi(0)$	0.3106	0.3095
$\pi(Z \times Z)$	0.3101	0.3095
$\pi(1, 1)$	0.0205	0.0209
$\pi(1, -1)$	0.0209	0.0209
$\pi(1, 0)$	0.1693	0.1694
$\pi(0, 1)$	0.1686	0.1694

and

$$\begin{aligned} \pi(0)|_{-1/\tau} &= \pi(Z\times Z)|_{-1/\tau} = \pi(0)|_{\tau} = \pi(Z\times Z)|_{\tau} \\ \pi(0)|_{\tau+1} &= \pi(Z\times Z)|_{\tau+1} = \pi(0)|_{\tau} = \pi(Z\times Z)|_{\tau} \end{aligned} \tag{3.18}$$

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